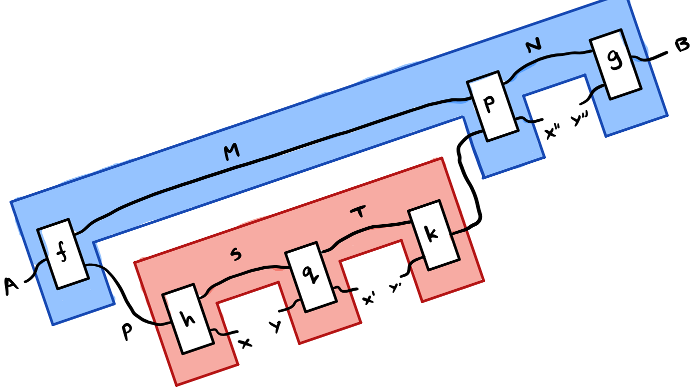

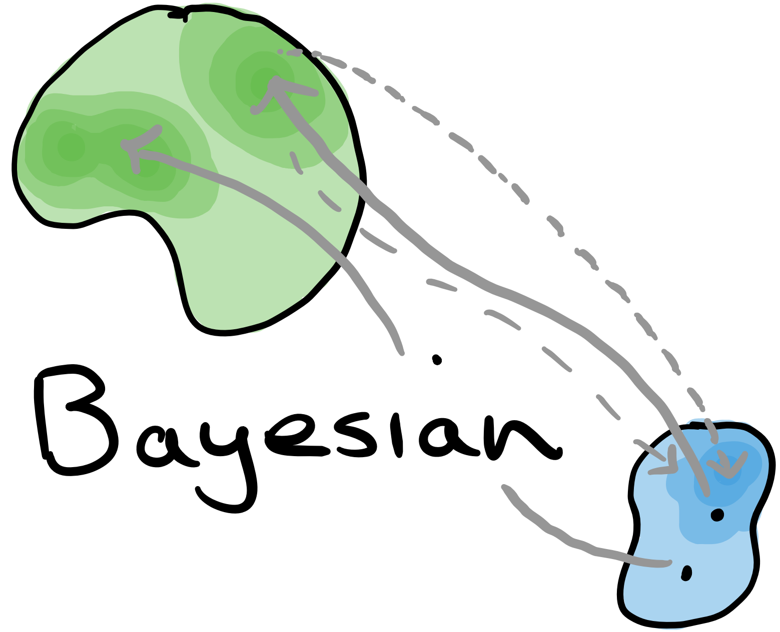

Compositional Bayesian Inference and Stochastic Combs

Dylan Braithwaite

University of Strathclyde

Computing and Information Science

Bayesian Inversion

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Bayes law: Invert a conditional distribution to update a prior belief.

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Based on an example from 'Probabilistic Programs as Measures', Staton (2020).

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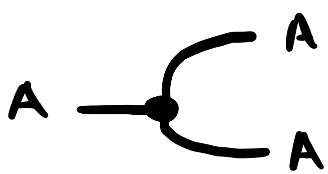

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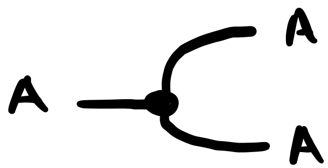

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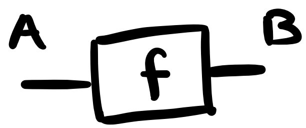
Morphisms 

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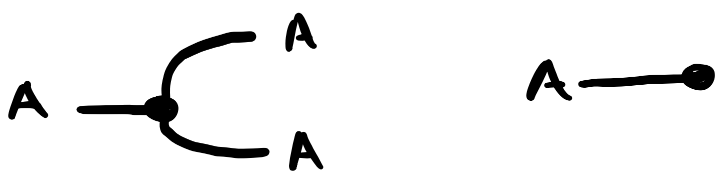

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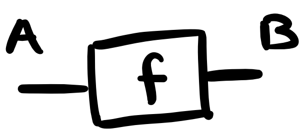

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"probability that a is mapped to b"

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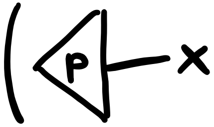
Internally, a distribution on X is a state $p : I \longrightarrow X$, () .

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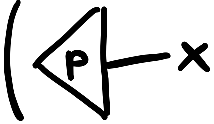
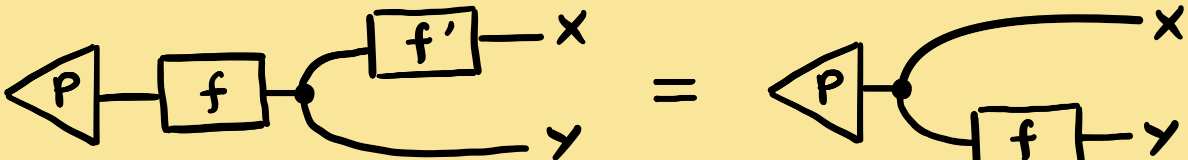
Internally, a distribution on X is a state $p: I \longrightarrow X$, ().

Defn (Cho & Jacobs, 2019)

A Bayesian inverse to $f: X \longrightarrow Y$ at p is a morphism $f': Y \longrightarrow X$ satisfying



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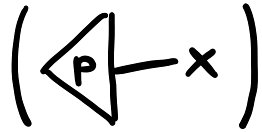
Internally, a distribution on X is a state $p : I \longrightarrow X$, () .

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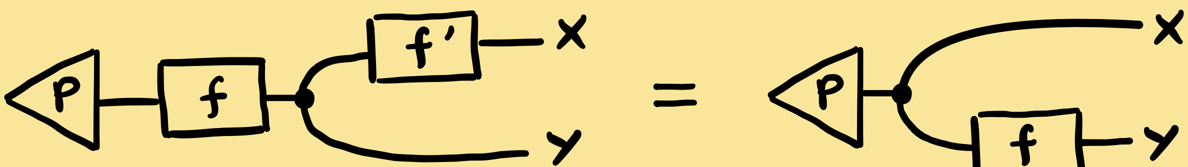

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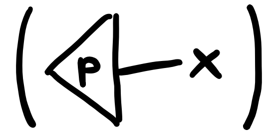
Internally, a distribution on X is a state $p : I \longrightarrow X$, .

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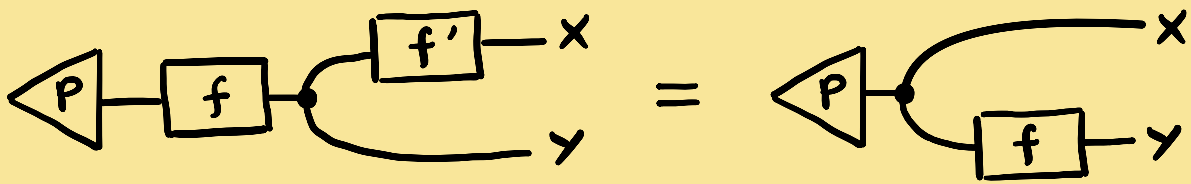

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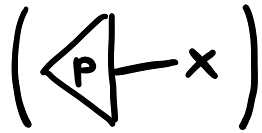
Internally, a distribution on X is a state $p : I \longrightarrow X$, () .

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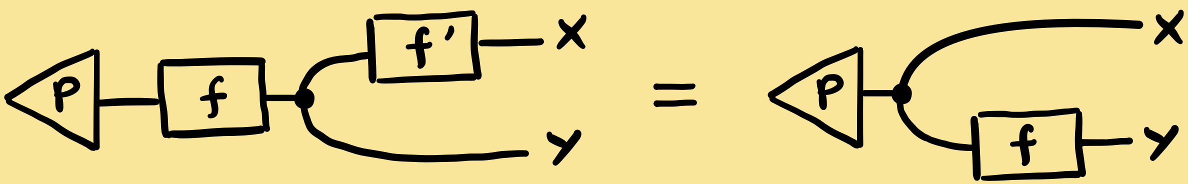

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- Not guaranteed to exist (but always does in many examples).

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- (Braithwaite, Smithe, Hedges, 2023) Fix functoriality using families of support objects.

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Can we represent inverses in a category where this makes sense?

Aside: Representable Markov Categories

(Fritz, Gonda, Perrone, Rischel 2023)

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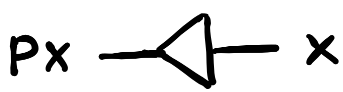
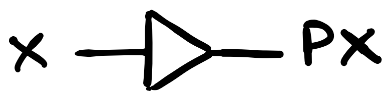


sampling morphism



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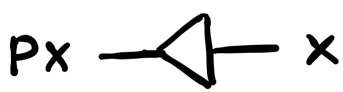

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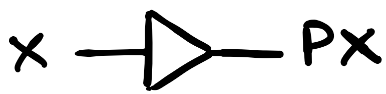

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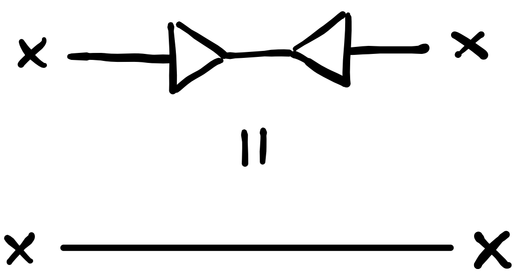

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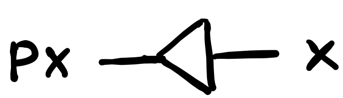

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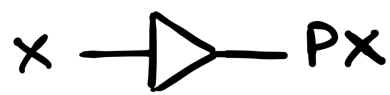

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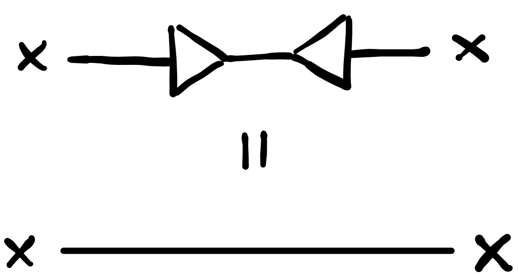

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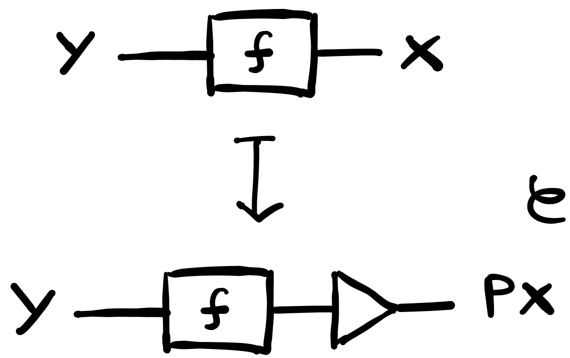

is an isomorphism $\mathcal{C}(Y,X) \cong \mathcal{C}_{det}(Y,PX)$

(Fritz, Gonda, Perrone, Rischel 2023)

Internally Parameterised Inverses

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Markov category with conditionals:

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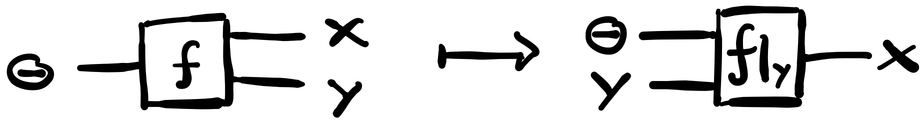

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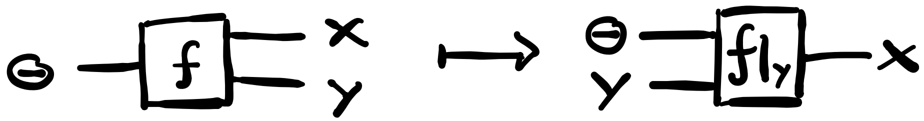

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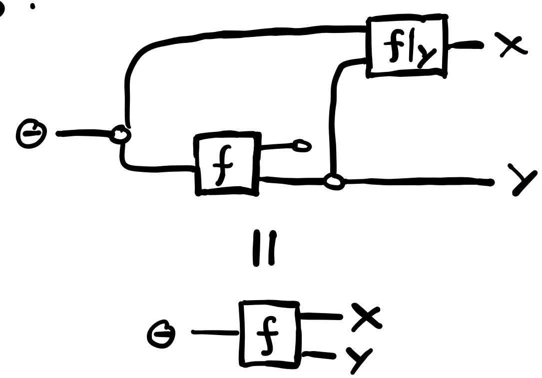

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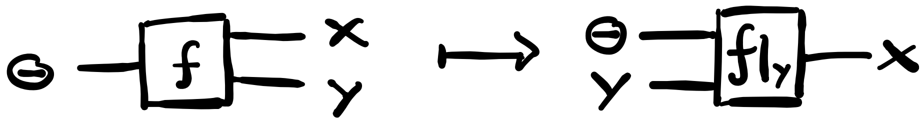

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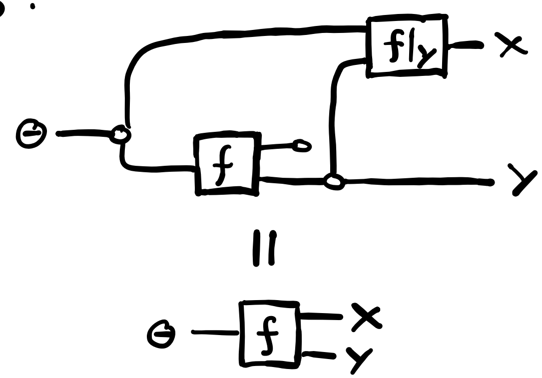
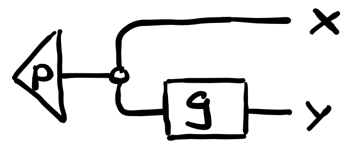


<u>Proposition</u> $g' : Y \to X$ is a conditional on Y of



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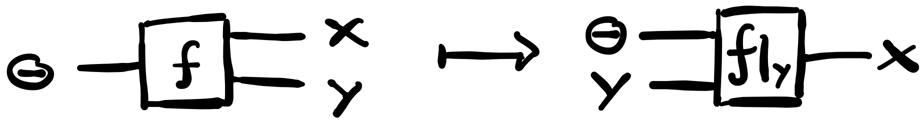

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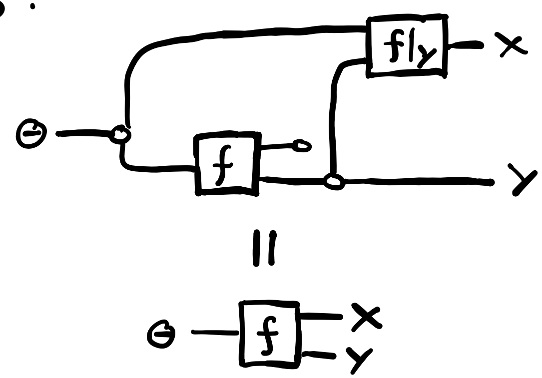
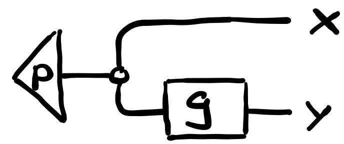


<u>Proposition</u> $g' : Y \to X$ is a conditional on Y of 

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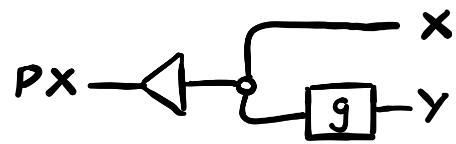
<u>Proposition</u> (Parametric Inverses)

Let $g^{\#} : PX \otimes Y \to X$ be a conditional on Y of 

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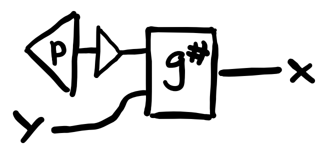
Then for all $p : I \to X$ we have that 

is a Bayesian inverse to g at p.

Composing Parametric Inverses

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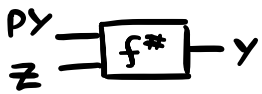
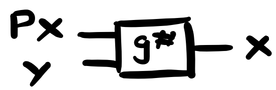
Let  ,  be 'parametric inverses' as defined in the previous proposition.

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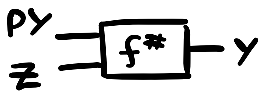
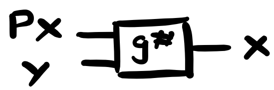
Let 

, 

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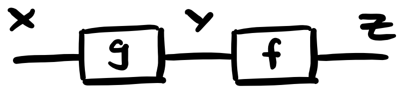
Q: Is the composite of $f^{\#}$ and $g^{\#}$ an inverse for 

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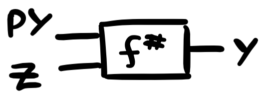
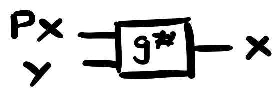
Let 

, 

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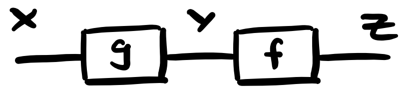
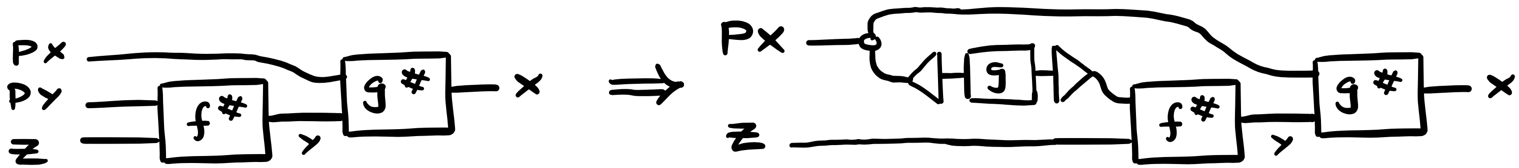
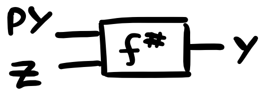
Q: Is the composite of $f^\#$ and $g^\#$ an inverse for 

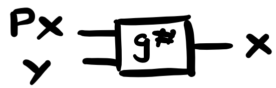
?

A: Yes, laxly.

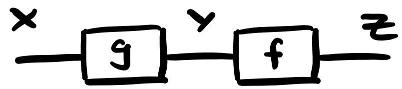


Composing Parametric Inverses

Let 

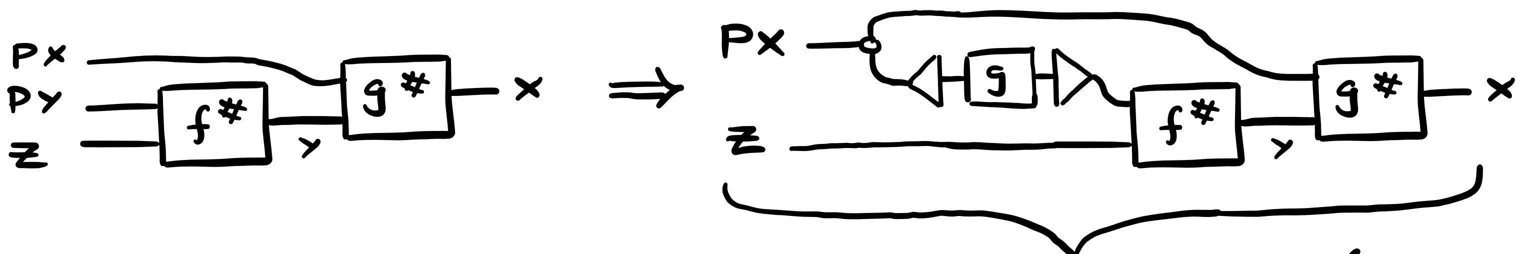
, 

be 'parametric inverses' as defined in the previous proposition.

Q: Is the composite of $f^{\#}$ and $g^{\#}$ an inverse for 

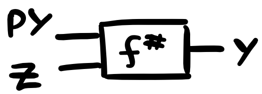
?

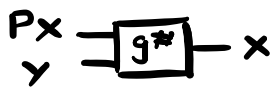
A: Yes, laxly.



parametric inverse to f;g

Composing Parametric Inverses

Let 

, 

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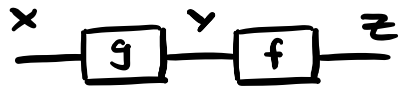
Q: Is the composite of $f^{\#}$ and $g^{\#}$ an inverse for 

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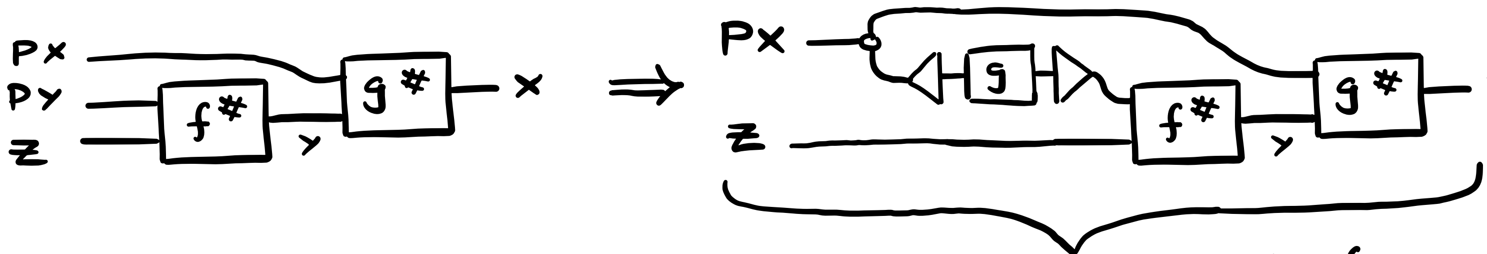
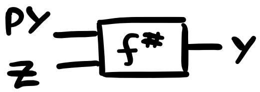
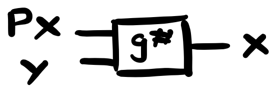


parametric inverse to f;g

$\text{Inv} : \mathcal{C}^{op} \longrightarrow \text{Para}(\mathcal{C})$ is 'almost surely' a lax functor

Composing Parametric Inverses

Let 

, 

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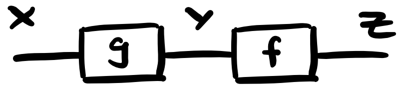
Q: Is the composite of $f^{\#}$ and $g^{\#}$ an inverse for 

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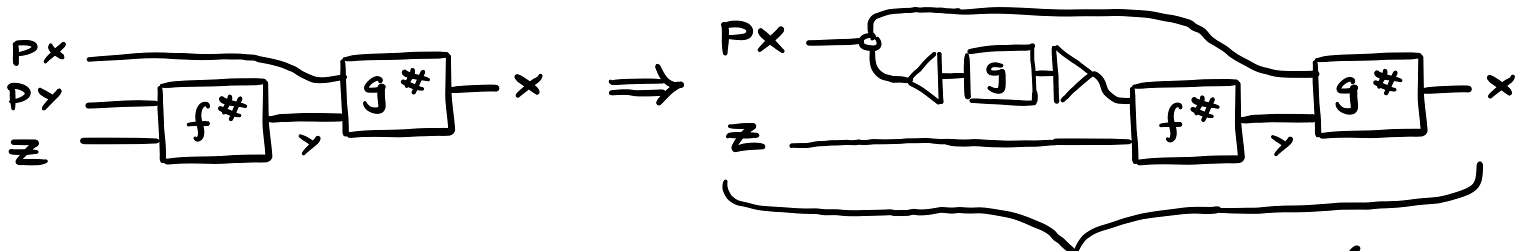


parametric inverse to $f;g$

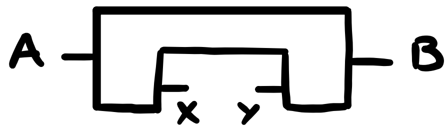
$\text{Inv} : \mathcal{C}^{op} \longrightarrow \text{Para}(\mathcal{C})$ is 'almost surely' a lax functor

Q: Why is this lax while the inversion functor $\mathcal{C} \longrightarrow \text{BLens}(\mathcal{C})$ was strict?

Stochastic Combs

Stochastic Combs

A 1-comb $\begin{pmatrix} A \\ B \end{pmatrix} \to \begin{pmatrix} X \\ Y \end{pmatrix}$ is



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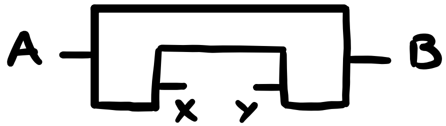
A 1-comb $\begin{pmatrix}A\\B\end{pmatrix} \longrightarrow \begin{pmatrix}X\\Y\end{pmatrix}$ is 

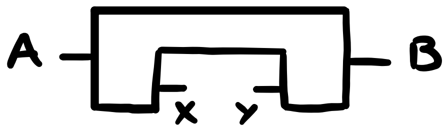
$$\mathrm{Comb}_1\left(\begin{pmatrix}A\\B\end{pmatrix};\begin{pmatrix}X\\Y\end{pmatrix}\right) = \int^{M} \mathcal{C}(A, M\otimes X)\times \mathcal{C}(M\otimes Y, B)$$

Stochastic Combs

AKA: "optics"

"monoidal lenses"

"doubles"

A 1-comb $\begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} X \\ Y \end{pmatrix}$ is 

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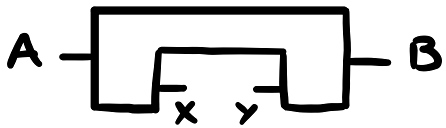
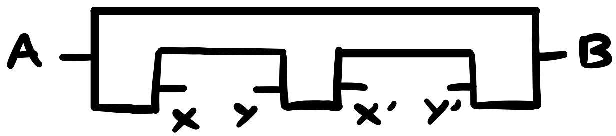


$$\mathrm{Comb}_1\left(\begin{pmatrix} A \\ B \end{pmatrix};\begin{pmatrix} X \\ Y \end{pmatrix}\right) = \int^M \mathcal{C}(A, M\otimes X)\times \mathcal{C}(M\otimes Y, B)$$

A 2-comb $\begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X' \\ Y' \end{pmatrix}$ is



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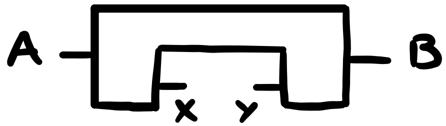

$$\mathrm{Comb}_1\left(\begin{pmatrix} A \\ B \end{pmatrix} ; \begin{pmatrix} X \\ Y \end{pmatrix}\right) = \int^{M} \mathcal{C}(A, M \otimes X) \times \mathcal{C}(M \otimes Y, B)$$

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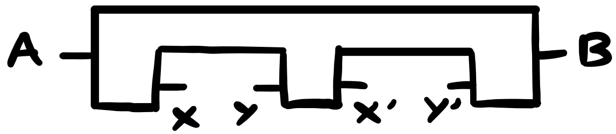

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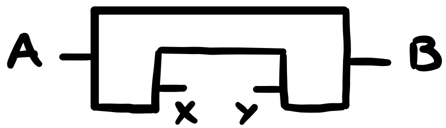

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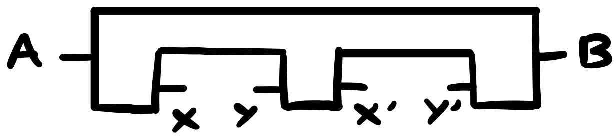

$$\mathrm{Comb}_2\left(\begin{pmatrix} A \\ B \end{pmatrix} ; \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X' \\ Y' \end{pmatrix}\right) = \int^{M,N} \mathcal{C}(A, M \otimes X) \times \mathcal{C}(M \otimes Y, N \otimes X') \times \mathcal{C}(N \otimes Y', B)$$

Stochastic Combs

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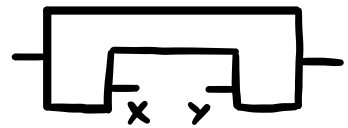
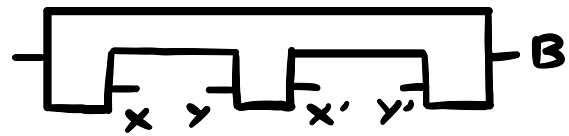
A 1-comb $\binom{A}{B} \to \binom{X}{Y}$ is



$$\mathrm{Comb}_1\left(\binom{A}{B};\binom{X}{Y}\right) = \int^{M} \mathcal{C}(A, M\otimes X)\times \mathcal{C}(M\otimes Y, B)$$

A 2-comb $\binom{A}{B} \to \binom{X}{Y}, \binom{X'}{Y'}$ is

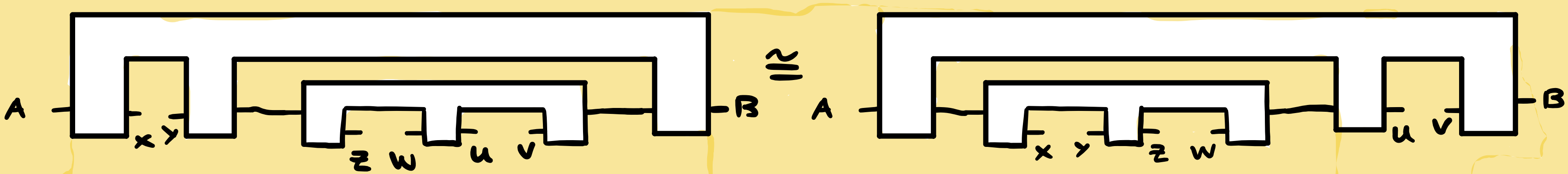


$$\mathrm{Comb}_2\left(\binom{A}{B};\binom{X}{Y},\binom{X'}{Y'}\right) = \int^{M,N} \mathcal{C}(A, M\otimes X)\times \mathcal{C}(M\otimes Y, N\otimes X')\times \mathcal{C}(N\otimes Y', B)$$

Define n-combs inductively by profunctor composition:

Earnshaw, Hefford & Román (CSL 2023):

Comb composition is a promonoidal product.



Bayesian Inverses as Combs

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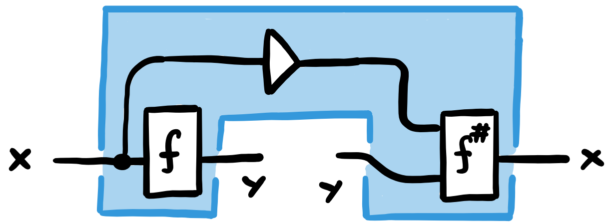
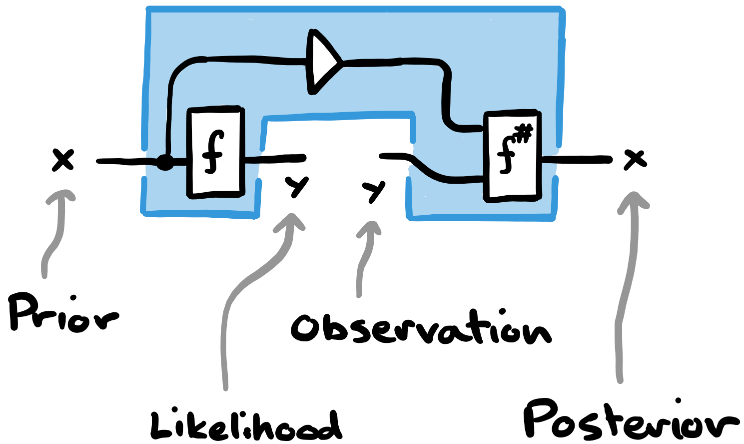


Bayesian Inverses as Combs



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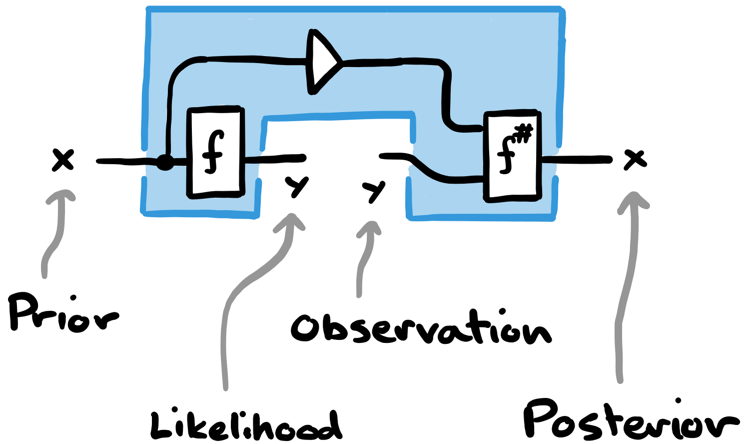
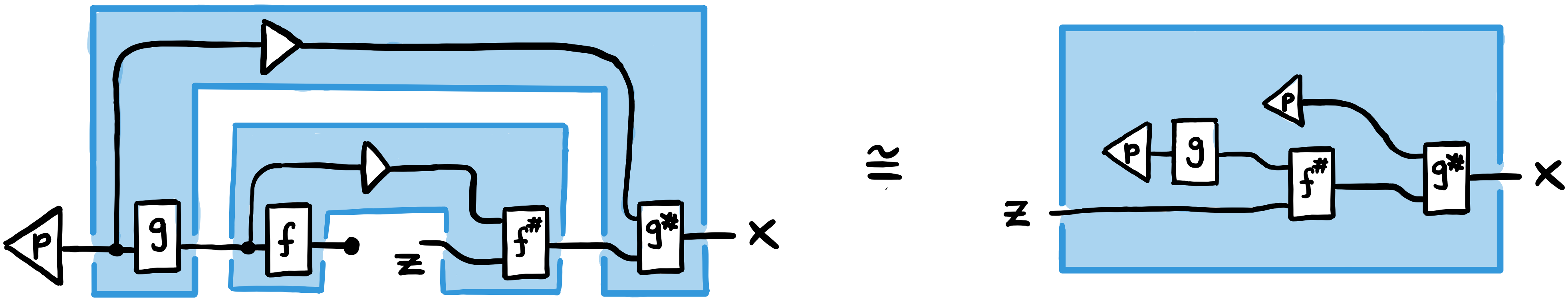


- Composite inverses are nested combs



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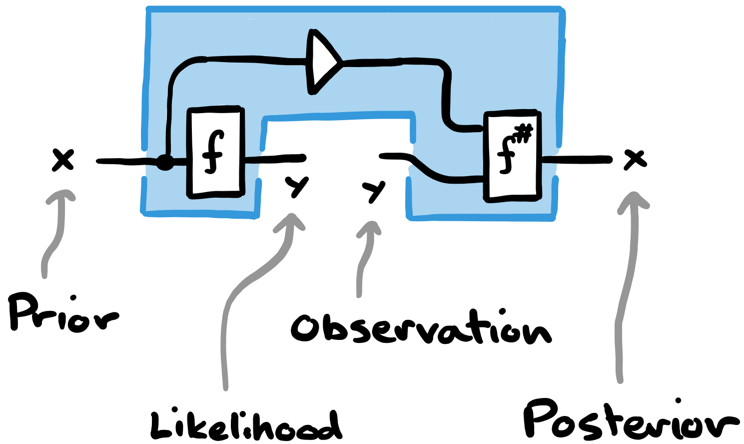

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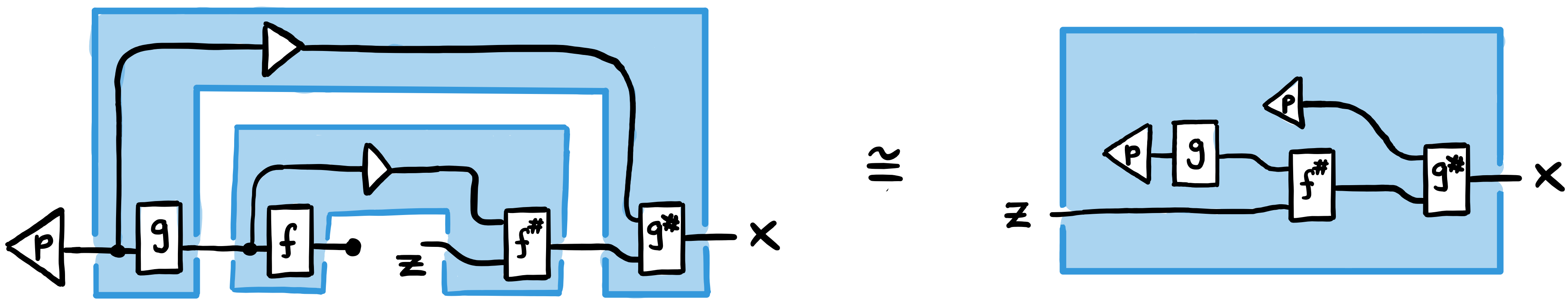
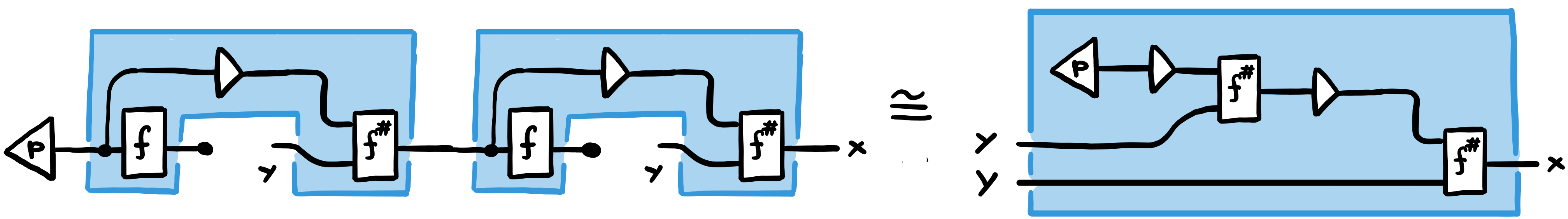


- Repeated updates are end-to-end composition



Example: Building statistical models from combs

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```
def example():
    let x := sample(Bernoulli(2/7))
        r := if x then 3 else 10
    in observe 4 from Poisson(r);
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Example from earlier.

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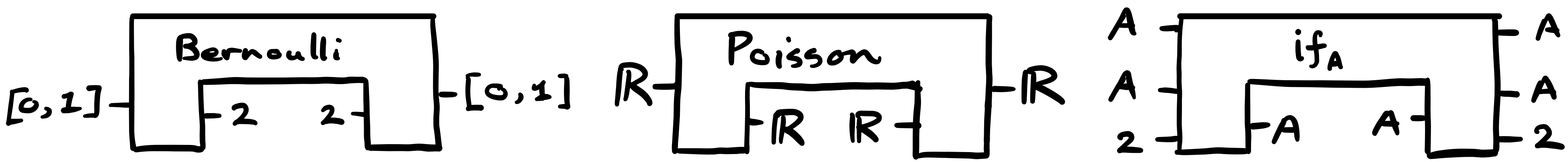

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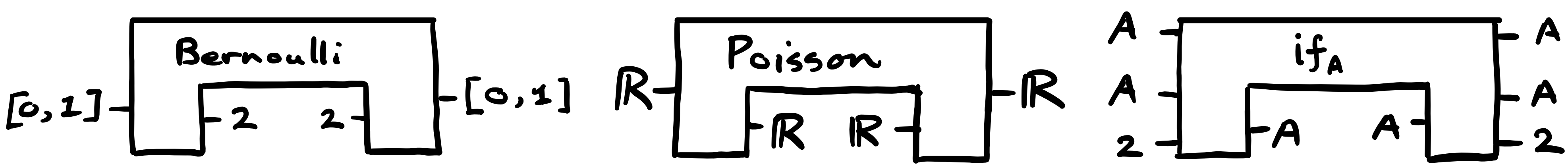

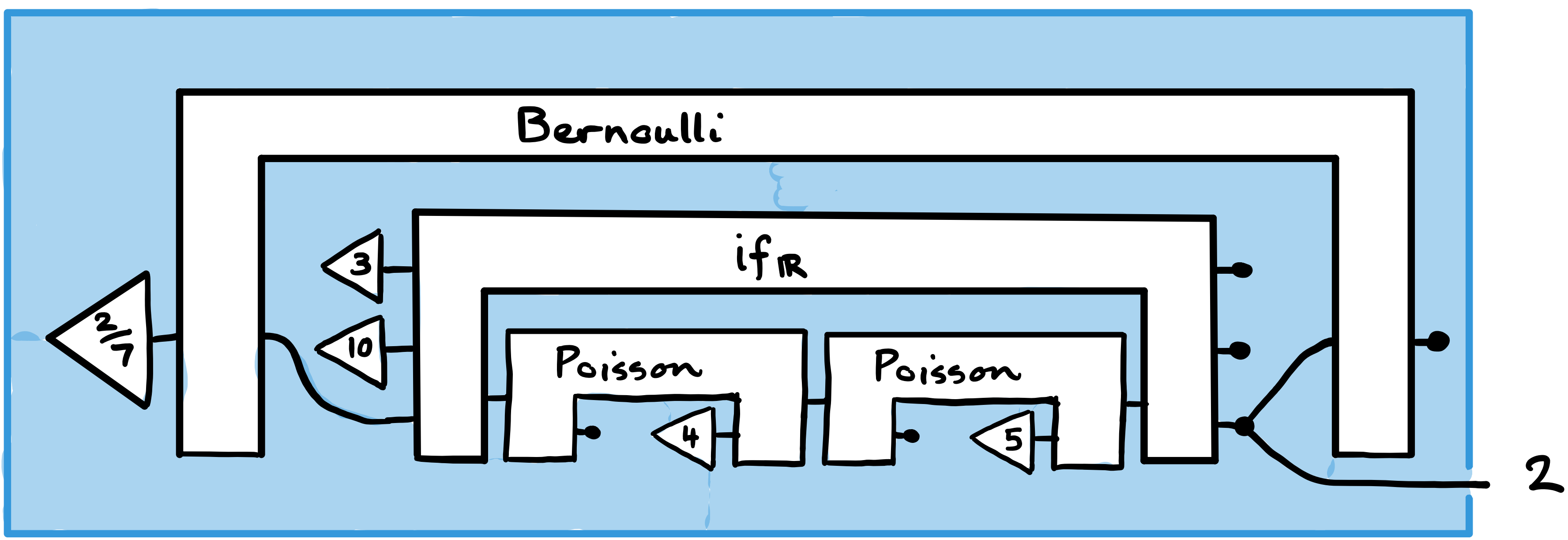

Combs as Models for Probabilistic Programs

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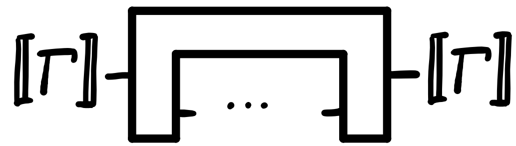
Expressions in context Γ are modelled as 

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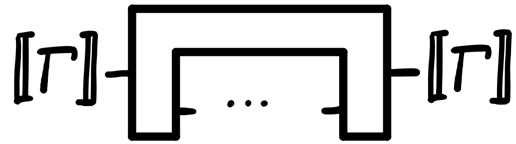
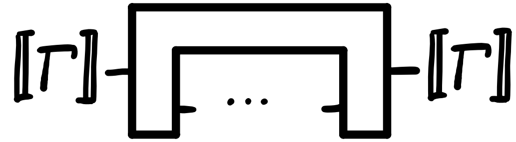


Three kinds of expression:

Combs as Models for Probabilistic Programs

Expressions in context Γ are modelled as 

Three kinds of expression:

- Typed terms $\quad \Gamma \vdash t : T$ 

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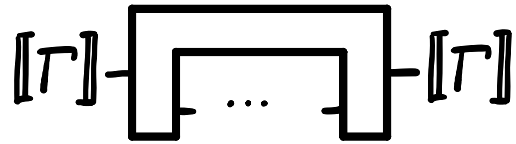
Expressions in context Γ are modelled as 

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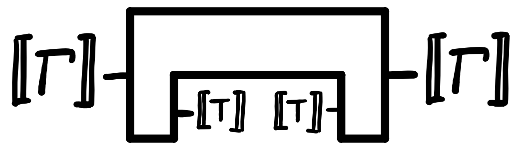
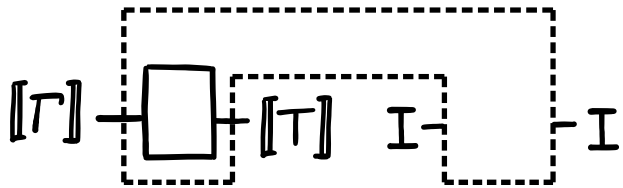


- Values $\Gamma \vdash_v t : T$



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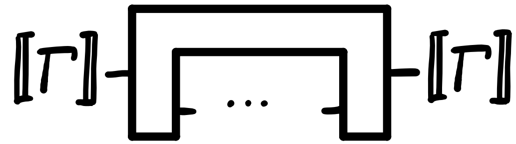
Expressions in context Γ are modelled as 

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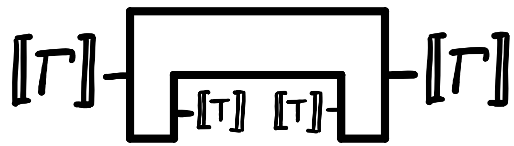
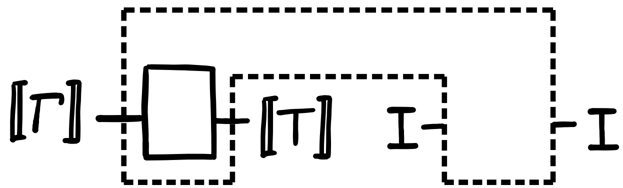
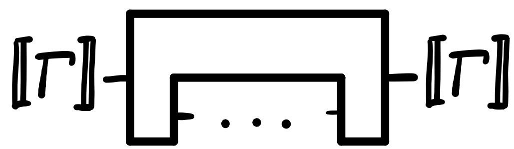
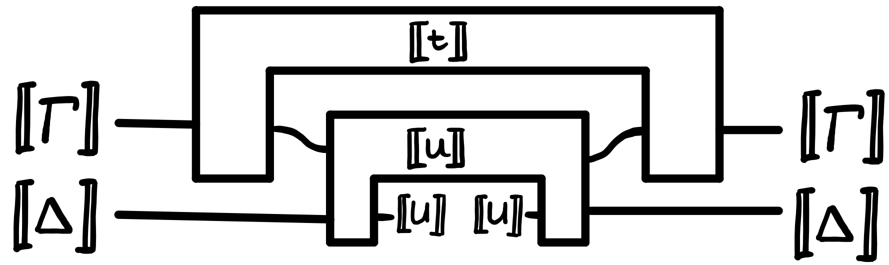
- Typed terms $\quad \Gamma \vdash t : T$ 

- Values $\quad \Gamma \vdash_v t : T$ 

- Statements $\quad \Gamma \vdash s \ \mathrm{stmt}$ 

$$\frac{\Gamma \vdash t : T \qquad \Delta, T \vdash u : U}{\Delta, \Gamma \vdash u[t] : U}$$



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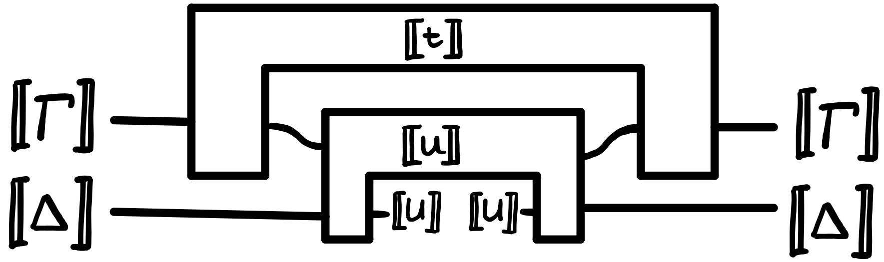
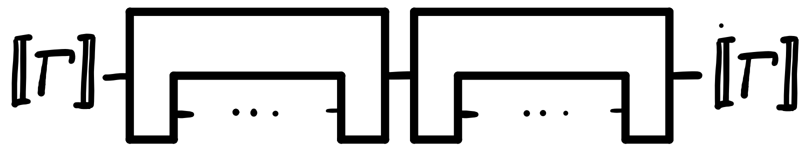


$$\frac{\Gamma \vdash s_1 \ \mathsf{stmt} \qquad \Gamma \vdash s_2 \ \mathsf{stmt}}{\Gamma \vdash s_1 ; s_2 \ \mathsf{stmt}}$$



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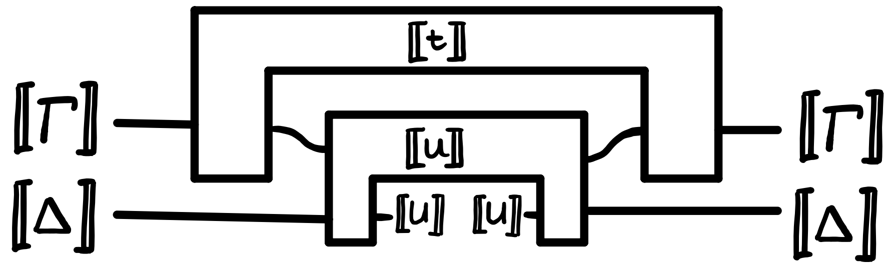

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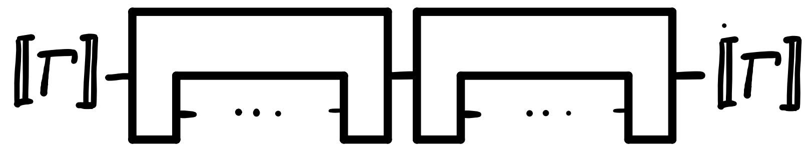
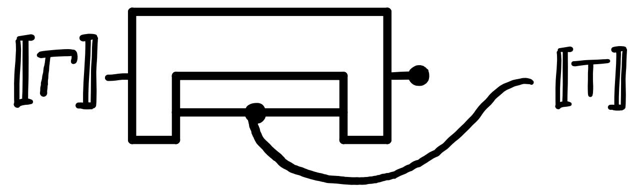
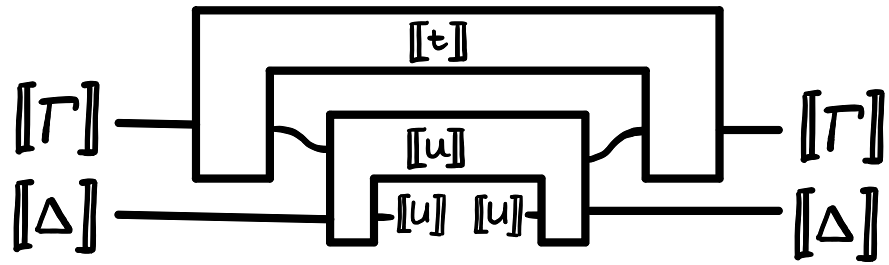


$$\frac{\Gamma \vdash t : T}{\Gamma \vdash_v \mathsf{sample}(t) : T}$$



$$\frac{\Gamma \vdash t : T \qquad \Delta, T \vdash u : U}{\Delta, \Gamma \vdash u[t] : U}$$



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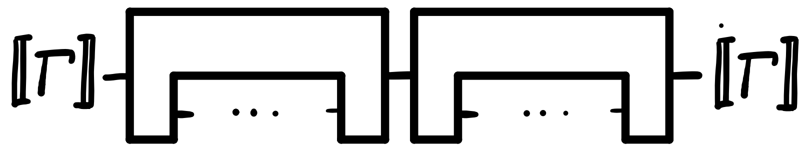

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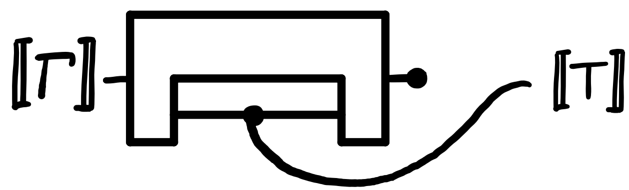
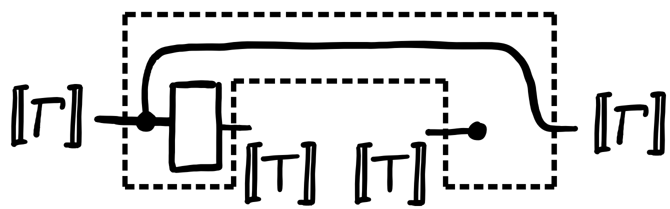
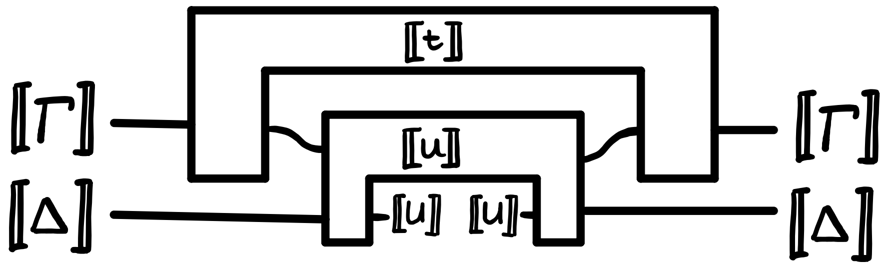


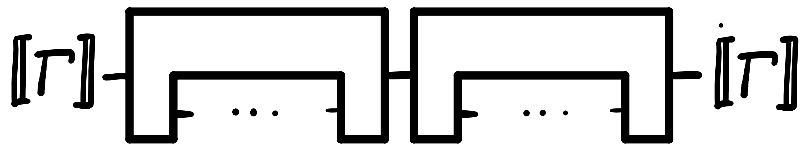
$$\frac{\Gamma \vdash_v t : T}{\Gamma \vdash \mathsf{certain}(t) : T}$$



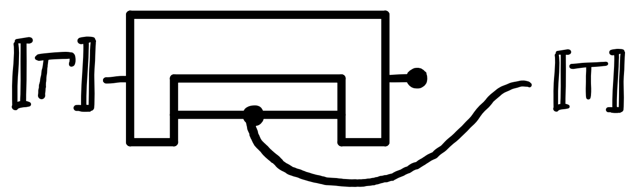
$$\frac{\Gamma \vdash t : T \qquad \Delta, T \vdash u : U}{\Delta, \Gamma \vdash u[t] : U}$$



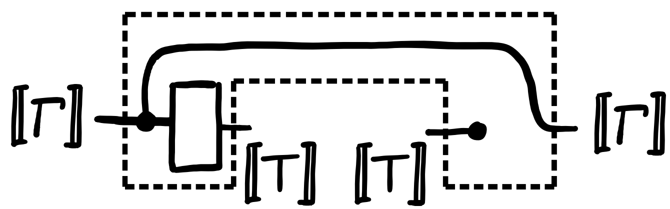
$$\frac{\Gamma \vdash s_1 \ \mathrm{stmt} \qquad \Gamma \vdash s_2 \ \mathrm{stmt}}{\Gamma \vdash s_1 ; s_2 \ \mathrm{stmt}}$$



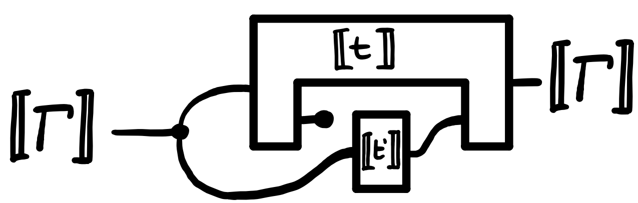
$$\frac{\Gamma \vdash t : T}{\Gamma \vdash_v \mathrm{sample}(t) : T}$$



$$\frac{\Gamma \vdash_v t : T}{\Gamma \vdash \mathrm{certain}(t) : T}$$



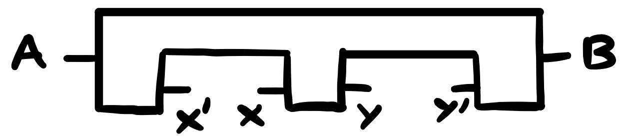
$$\frac{\Gamma \vdash t : T \qquad \Gamma \vdash_v t' : T}{\Delta, \Gamma \vdash (\mathrm{observe}\ t' \ \mathrm{from}\ t) \ \mathrm{stmt}}$$



Other thoughts

Other thoughts

- Higher order terms :



behaves like a 'curried comb' $\begin{pmatrix}A\\B\end{pmatrix} \longrightarrow \left[\begin{pmatrix}x\\y\end{pmatrix} \multimap \begin{pmatrix}x'\\y'\end{pmatrix}\right]$.

Other thoughts

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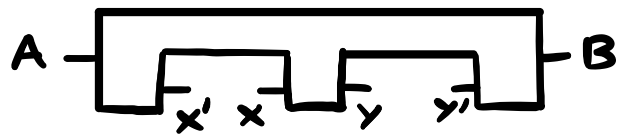
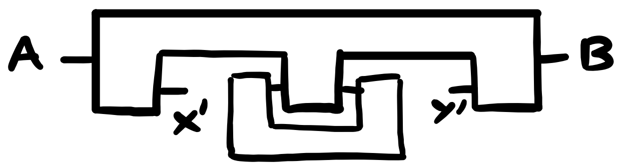


behaves like a 'curried comb' $\begin{pmatrix}A\\B\end{pmatrix} \longrightarrow \left[\begin{pmatrix}X\\Y\end{pmatrix} \multimap \begin{pmatrix}X'\\Y'\end{pmatrix}\right]$.

Eval :



Other thoughts

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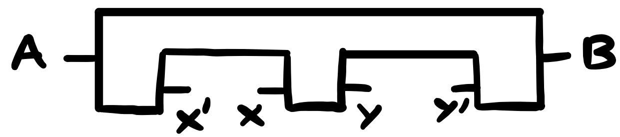

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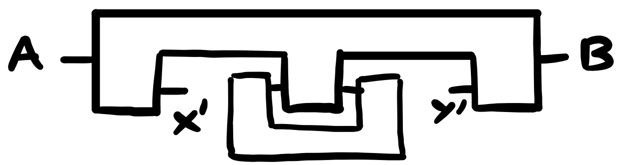
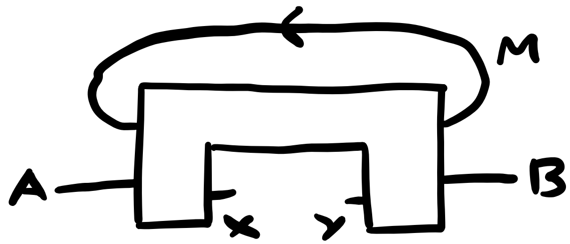


- Iteration ?



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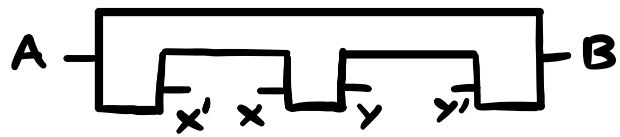

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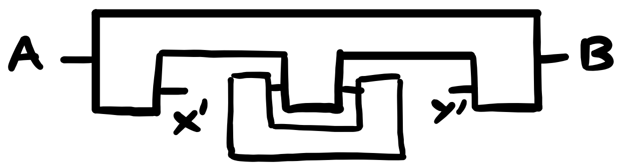

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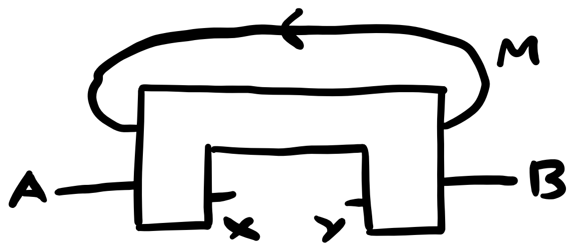
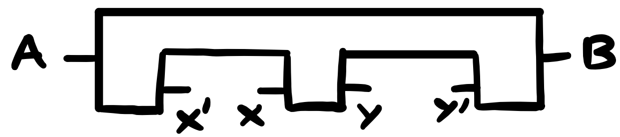


- Holes as models of I/O ?

Other thoughts

- Higher order terms : 

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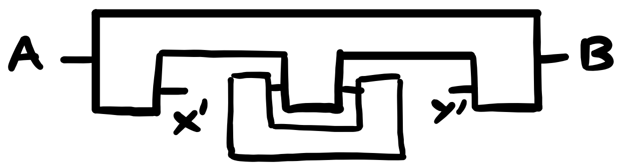
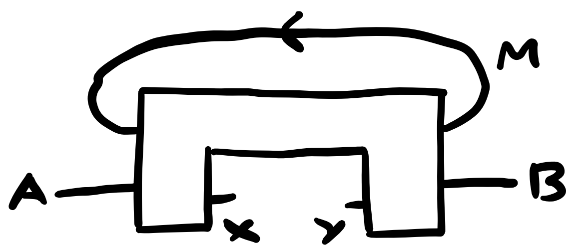
  Eval : 

- Iteration ? 

- Holes as models of I/O ?

  Can something akin to session types give a more first-class way to manipulate holes ?

Beyond representable Markov categories?

Beyond representable Markov categories?

Recall: Externally parameterized inverses in BLens

Beyond representable Markov categories?

Recall: Externally parameterized inverses in BLens

$$\mathrm{BLens}\left(\begin{matrix} A \\ X \end{matrix}, \begin{matrix} B \\ Y \end{matrix}\right) \cong \mathrm{Optic}_{\times, \pitchfork}\left(\begin{matrix} \text{よ}^{A} \\ \text{よ}_{x} \end{matrix}, \begin{matrix} \text{よ}^{B} \\ \text{よ}_{x} \end{matrix}\right)$$

Beyond representable Markov categories?

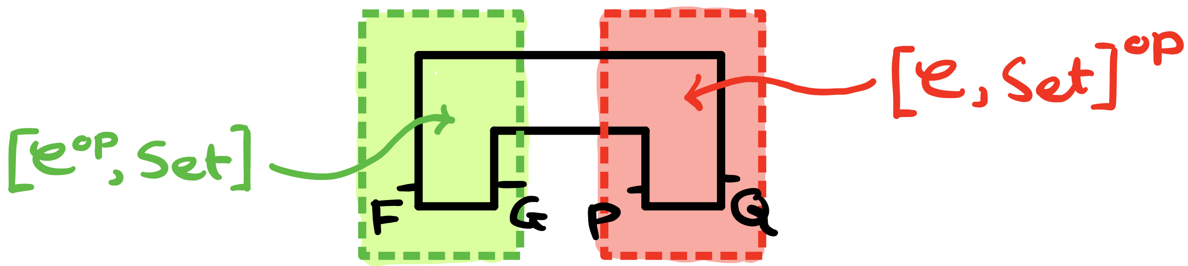
Recall : Externally parameterized inverses in BLens

$$\mathrm{BLens}\begin{pmatrix} A & B \\ X\ , & Y \end{pmatrix} \cong \mathrm{Optic}_{\times,\pitchfork}\begin{pmatrix} よ^{A} & よ^{B} \\ よ_{X}\ , & よ_{X} \end{pmatrix}$$

Mixed optics :

- Left category $[\mathcal{C}^{op}, \mathrm{Set}]$
- Right category $[\mathcal{C}, \mathrm{Set}]^{op}$

} connected by monoidal actions of $[\mathcal{C}^{op}, \mathrm{Set}]$



Beyond representable Markov categories?

Recall: Externally parameterized inverses in BLens

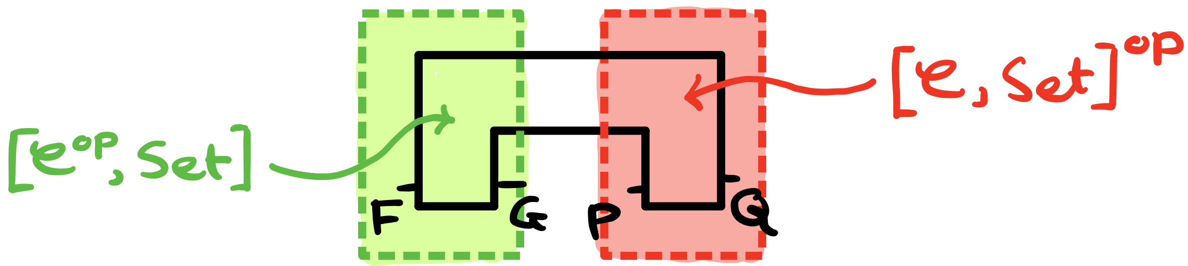
$$\mathbf{BLens}\left(\begin{pmatrix}A & B\\ X, & Y\end{pmatrix}\right) \cong \mathbf{Optic}_{\times,\pitchfork}\left(\begin{pmatrix}\mathcal{K}^{A} & \mathcal{K}^{B}\\ \mathcal{K}_{X}, & \mathcal{K}_{X}\end{pmatrix}\right)$$

Mixed optics:

- Left category $[\mathcal{C}^{op}, \mathbf{Set}]$
- Right category $[\mathcal{C}, \mathbf{Set}]^{op}$

connected by monoidal actions of $[\mathcal{C}^{op}, \mathbf{Set}]$

$$\begin{pmatrix}F\\P\end{pmatrix} \longrightarrow \begin{pmatrix}G\\Q\end{pmatrix} \sim \int^{R\in\hat{\mathcal{C}}} \hat{\mathcal{C}}(F, R\times G) \times \check{\mathcal{C}}^{op}(Q^{R(I)}, P)$$



Beyond representable Markov categories?

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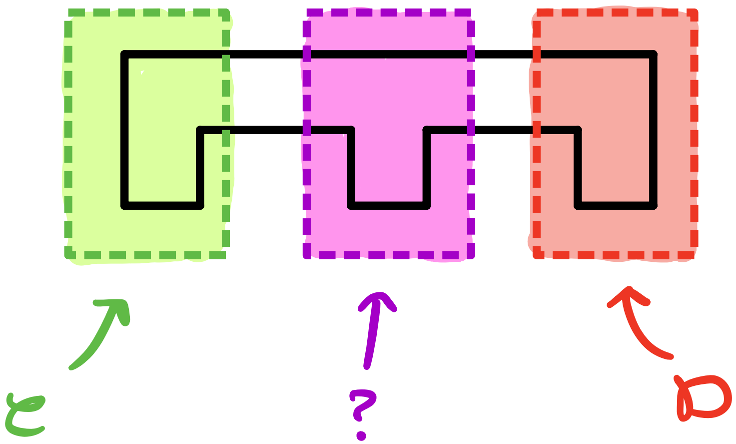
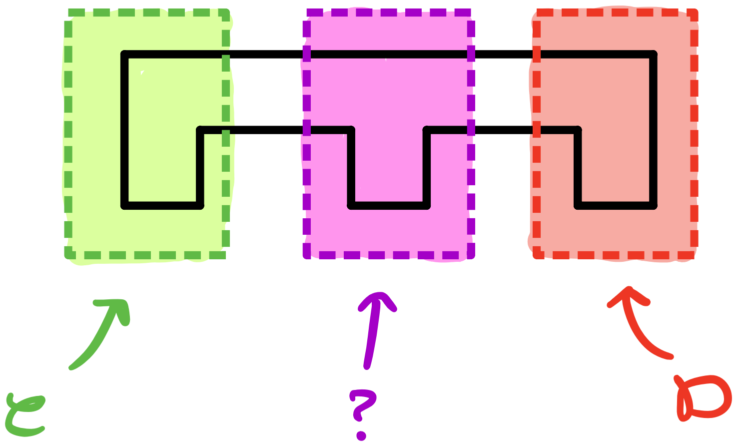


- Need a notion of heteromorphism $\mathcal{D} \nrightarrow \mathcal{C}$

Beyond representable Markov categories?



- Need a notion of heteromorphism $\mathcal{D} \nrightarrow \mathcal{C}$
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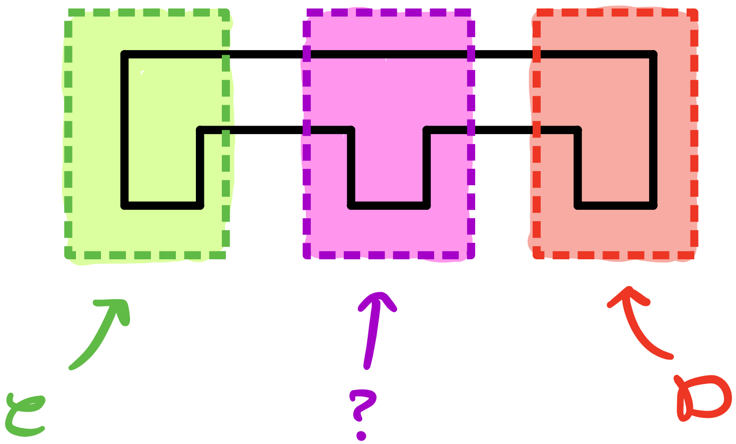
Beyond representable Markov categories?



- Need a notion of heteromorphism $\mathcal{D} \nrightarrow \mathcal{C}$
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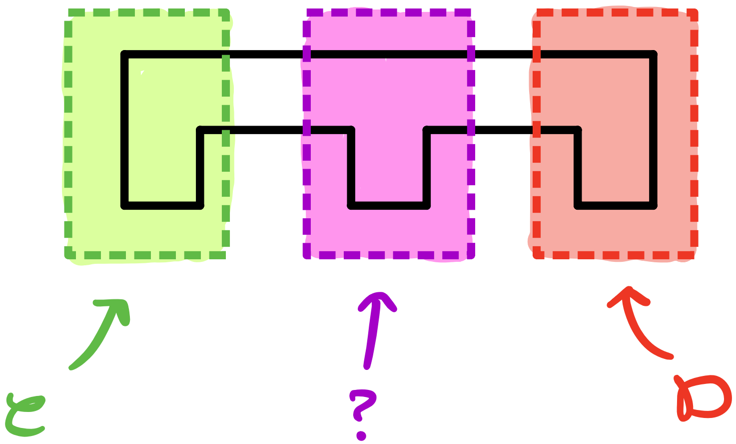
- With suitable operations for 'filling holes' in diagrams.

Beyond representable Markov categories?



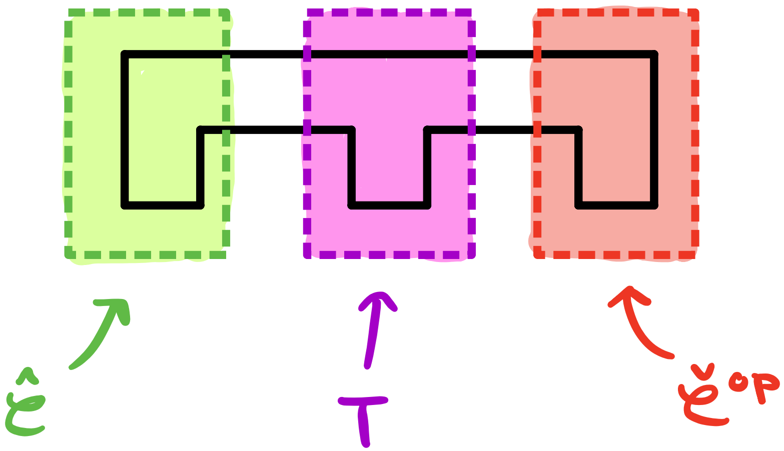
- Need a notion of heteromorphism $\mathcal{D} \nrightarrow \mathcal{C}$
- Compatible with the monoidal actions
- With suitable operations for 'filling holes' in diagrams.

<u>Claim</u>: If $T : \mathcal{D} \nrightarrow \mathcal{C}$ is an equivalence in Tamb (the bicategory of Tambara modules) then

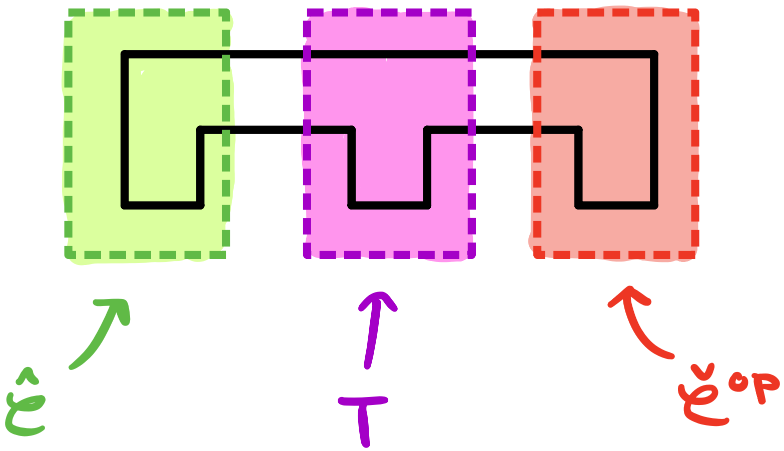
$$\begin{pmatrix} A \\ X \end{pmatrix}, \begin{pmatrix} B \\ Y \end{pmatrix}, \begin{pmatrix} C \\ Z \end{pmatrix} \longmapsto \int^{M,N} \mathcal{C}(A, M \bullet B) \times T(M \bullet Y, N \bullet C) \times \mathcal{D}(N \bullet Z, X)$$

defines a promonoidal product.

Beyond representable Markov categories?

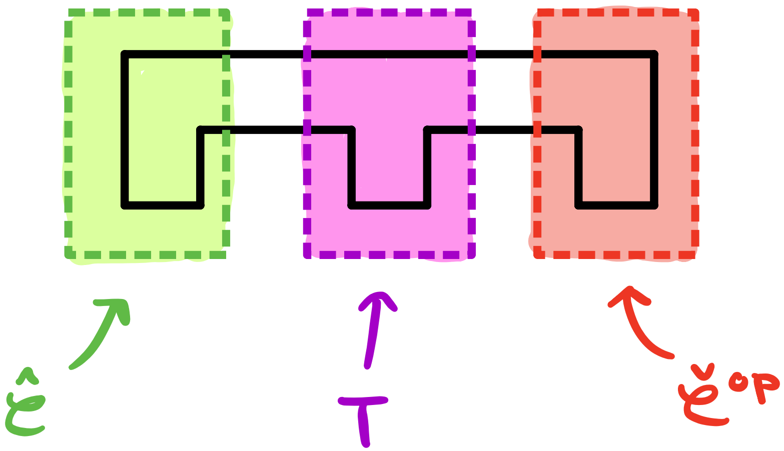


Beyond representable Markov categories?



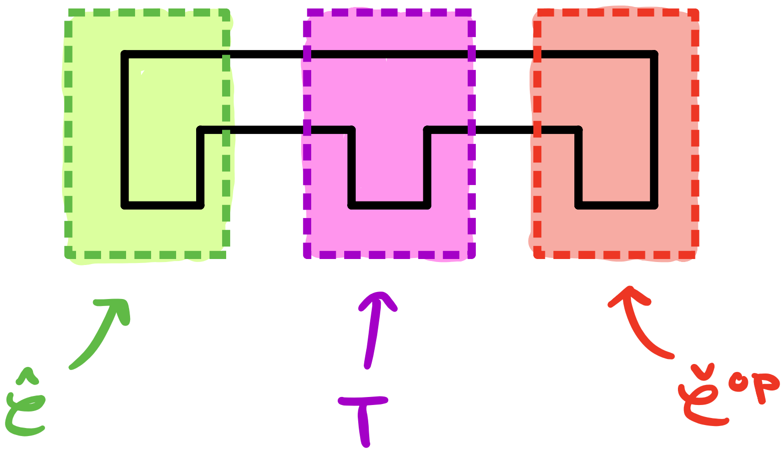
- Need a notion of heteromorphism $\check{\mathcal{C}}^{op} \nrightarrow \hat{\mathcal{C}}$

Beyond representable Markov categories?



- Need a notion of heteromorphism $\check{\mathcal{C}}^{op} \nrightarrow \hat{\mathcal{C}}$
- An equivalence in Tamb

Beyond representable Markov categories?



$$T : \check{\mathcal{C}} \times \hat{\mathcal{C}} \longrightarrow \mathrm{Set}$$

- Need a notion of heteromorphism $\check{\mathcal{C}}^{op} \nrightarrow \hat{\mathcal{C}}$
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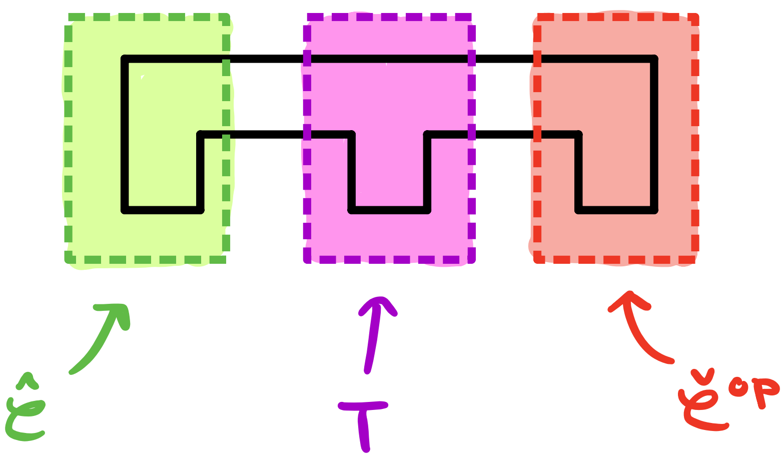
Beyond representable Markov categories?



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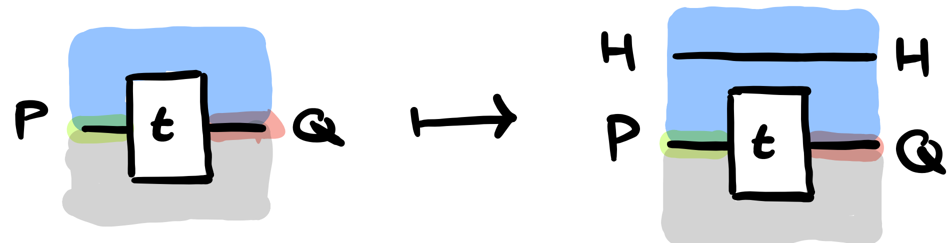
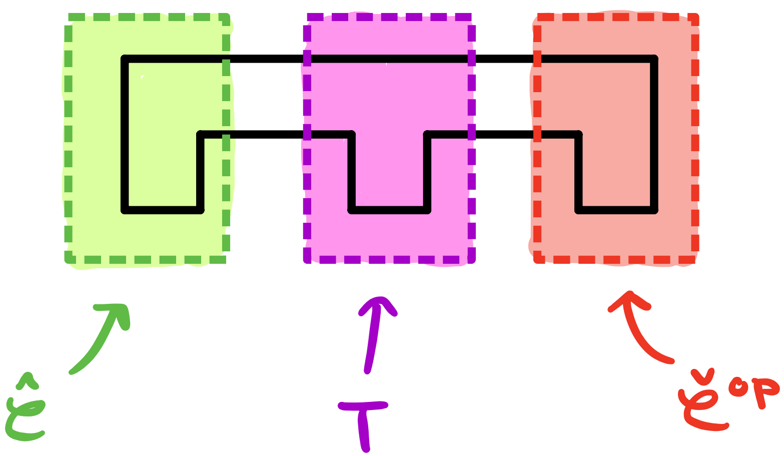


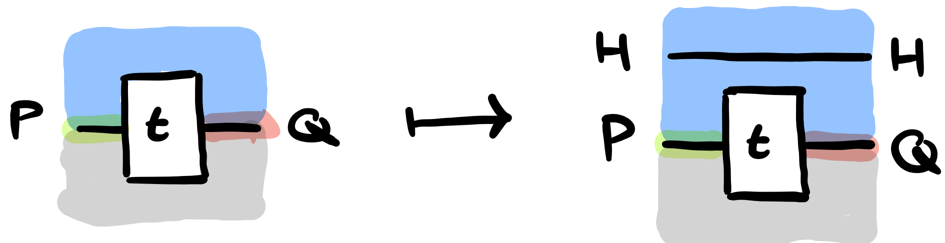
with a strength:

$$T(P,Q) \longrightarrow T(P^{H(I)}, H \times Q)$$

Beyond representable Markov categories?



- Need a notion of heteromorphism $\check{\mathcal{E}}^{op} \nrightarrow \hat{\mathcal{E}}$
- An equivalence in Tamb



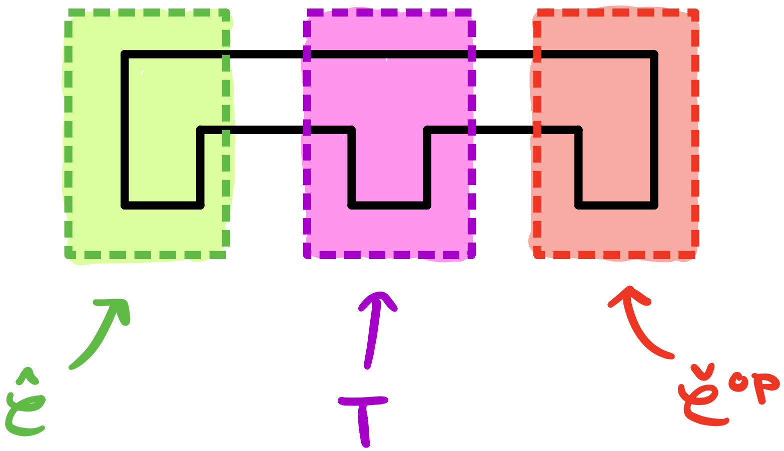
$$T : \check{\mathcal{E}} \times \hat{\mathcal{E}} \longrightarrow \mathrm{Set}$$

with a strength:

$$T(P, Q) \longrightarrow T(P^{H(I)}, H \times Q)$$

- same variance on P and Q, but different variance on actions

Beyond representable Markov categories?



- Need a notion of heteromorphism $\check{\mathcal{C}}^{op} \nrightarrow \hat{\mathcal{C}}$
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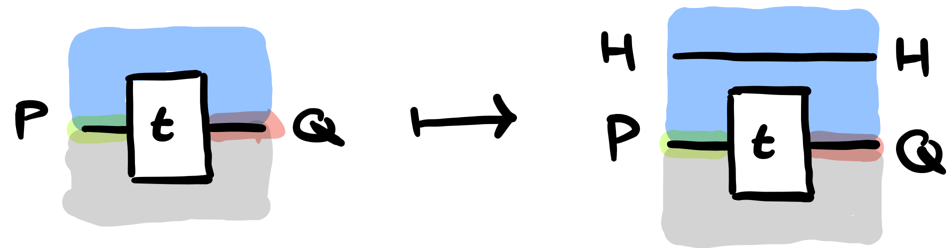
$$T : \check{\mathcal{C}} \times \hat{\mathcal{C}} \longrightarrow \mathrm{Set}$$



with a strength:

$$T(P, Q) \longrightarrow T(P^{H(I)}, H \times Q)$$

- same variance on P and Q, but different variance on actions
- Conjecture: No such structure possible

Thank you!

slides at : dylanbraithwaite.github.io